

Recency-based preferential attachment models

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Preferential attachment models were shown to be very effective in predicting such important properties of real-world networks as the power-law degree distribution, small diameter etc. However, they do not allow to model the so-called *recency property*. Recency property reflects the fact that in many real networks vertices tend to connect to other vertices of similar age. This fact motivated us to introduce and analyse a new class of models—recency-based models. This class is a generalization of fitness models, which were suggested by Bianconi and Barabási. Bianconi and Barabási extended preferential attachment models with pages' inherent quality or *fitness* of vertices. To additionally reflect a recency property, it is reasonable to generalize fitness models by adding a recency factor to the attractiveness function. This means that pages are gaining incoming links according to their *attractiveness*, which is determined by the incoming degree of the page (current popularity), its *inherent quality* (some page-specific constant) and age (new pages are gaining new links more rapidly). In this paper, we rigorously analyse the degree distribution in the most realistic recency-based model. Also, we prove that this model does reflect the recency property.

Keywords: random graph models; recency property; preferential attachment; power-law degree distribution.

1. Introduction

Numerous models have been suggested to reflect and predict the growth of the Web [1–3], the most well-known ones are preferential attachment models. One of the first attempts to propose a realistic mathematical model of the Web growth was made in [4]. The main idea is to start with the assumption that new pages often link to old popular pages. Barabási and Albert defined a graph construction stochastic process, which is a Markov chain of graphs, governed by the *preferential attachment*. At each step in the process, a new vertex is added to the graph and is joined to m different vertices already existing in the graph that are chosen with probabilities proportional to their incoming degree (the measure of popularity). This model successfully explained some properties of the Web graph like its small diameter and power-law distribution of incoming degrees. Later, many modifications to the Barabási–Albert model have been proposed, e.g. [5–7], in order to more accurately depict these but also other properties (see [8,9] for details).

It was noted by Bianconi & Barabási [10] that in real networks some vertices are gaining new incoming links not only because of their incoming degree (popularity), but also because of their own intrinsic

properties. Motivated by this observation, Bianconi and Barabási extended preferential attachment models with pages' inherent quality or *fitness* of vertices. When a new vertex is added to the graph, it is joined to some already existing vertices that are chosen with probabilities proportional to the product of their fitness and incoming degree.

One of the main drawbacks of these models is that they pay too much attention to old pages and do not realistically explain how links pointing to newly created pages appear. For example, most new media pages like news and blog posts are popular only for a short period of time, i.e. such pages are mostly cited and visited for several days after they appeared. In [11], a *recency property* was introduced, which reflects the fact that new media pages tend to connect to other media pages of similar age. Namely, for the media related part of the Web it was shown that $e(T)$ —the fraction of edges connecting the pages whose age difference is greater than T —decreases exponentially fast.

Although preferential attachment models reflect some important properties of real-world networks, they do not allow to model the recency property. Here we discuss recency-based models—a generalization of fitness models, where a recency factor is added to the attractiveness function. This means that pages are gaining incoming links according to their *attractiveness*, which is determined by the incoming degree of the page (current popularity), its *inherent quality* (some page-specific constant) and age (new pages are gaining new links more rapidly).

The idea of adding the recency factor to the attractiveness function was first suggested in [11]. In this paper, we propose a new formal definition of the model from [11]. Also, this rigorous definition allows us to theoretically analyse different properties of the recency-based models more thoroughly using a combinatorial approach. Our analysis shows that for the considered models the power-law distribution of inherent quality leads to the power-law degree distribution. We also rigorously analyse the recency property, i.e. the behaviour of $e(T)$.

2. Motivation

In [11], a model of the Media Web evolution has been proposed. The idea was to generalize the preferential attachment in the way that the probability to cite a page p is proportional to the *attractiveness* of p , which is some function of $d(p)$ (current degree of p), $q(p)$ (intrinsic quality of p), and $a(p)$ (current age of p). Different attractiveness functions were considered in [11]:

$$\text{attr}(p) = q(p)^{\alpha_1} \cdot d(p)^{\alpha_2} \cdot e^{-(a(p)/\tau) \cdot \alpha_3},$$

where $(\alpha_1, \alpha_2, \alpha_3) \in \{0, 1\}^3$ and τ corresponds to the mean lifetime of the decaying attractiveness. For example, $\text{attr}(p) = d(p)$ leads to preferential attachment, while $\text{attr}(p) = q(p) \cdot d(p)$ leads to fitness model.

To depict the recency property of the Media Web one has to include the recency factor $e^{-a(p)/\tau}$ in the attractiveness function. Further mean-field approximation analysis and computer simulations [11] showed that in order to have the power-law with a realistic exponent, attractiveness function $\text{attr}(p) = q(p) e^{-a(p)/\tau}$ should be chosen. Moreover, the distribution of qualities q should follow the power law. That was also confirmed by the analysis of the likelihood of a real data given the model.

Note that some other recency factors have been previously proposed. In [12,13], attractiveness function $\text{attr}(p) = d(p) \cdot a(p)^{-\alpha}$ was studied using the mean-field approximation and computer simulations. It was shown that the degree distribution follows the power law for $\alpha < 1$ and is exponential for $\alpha > 1$. Of course, it is also interesting to analyse $\text{attr}(p) = d(p) \cdot a(p)^{-\alpha}$ in a mathematically rigorous way, but this is beyond the scope of this paper.

Thus, in this paper we mostly focus on the attractiveness function $q(p) \cdot e^{-a(p)/\tau}$ and the goal is to formalize this model and then analyse it rigorously. In addition, our analysis allows to understand under which assumptions the conclusions made in [11] hold.

3. Model

In this section, we formalize the model introduced in [11]. We construct a sequence of random graphs $\{G_n\}$. This sequence has the following parameters: a positive integer constant m (vertex outdegree) and an integer function $N(n)$. We also need a sequence of mutually independent random variables ζ_1, ζ_2, \dots with some given distribution taking positive values. Each graph G_n is defined according to its own constructing procedure which is based on the idea of preferential attachment.

Let us now define the random graph G_n . At the beginning of the constructing process we have two vertices and one edge between them (graph \tilde{G}_2^n). The first two vertices have inherent qualities $q(1) := \zeta_1$ and $q(2) := \zeta_2$. At the $(t + 1)$ th step ($2 \leq t \leq n - 1$) one vertex and m edges are added to \tilde{G}_t^n . New vertex $t + 1$ has an inherent quality $q(t + 1) := \zeta_{t+1}$. New edges are drawn independently and they go from the new vertex to previous vertices. For each edge, the probability that it goes to a vertex i ($1 \leq i \leq t$) is equal to

$$\frac{\text{attr}_t(i)}{\sum_{j=1}^t \text{attr}_t(j)},$$

where

$$\text{attr}_t(i) = q(i) e^{-(t-i)/N(n)}.$$

According to the definition, loops are not allowed, although multiple edges may appear.

It is important to note that, in contrast to standard definitions of preferential attachment models, in our case a graph G_n cannot be obtained from a graph G_{n-1} . Each graph has its own constructing procedure which is based on preferential attachment. The reason is that, in contrast to [11], the mean lifetime of the decaying attractiveness $N(n)$ varies with n . This unusual definition allows us to rigorously analyse both the power-law degree distribution and the behaviour of $e(T)$, which is the fraction of edges connecting vertices i and j with $|i - j| > T$. Furthermore, we assume that $N(n) \rightarrow \infty$ as n grows. This allows us to analyse the fraction of vertices of degree $d = d(n)$ when $d(n)$ may grow with n .

The choice of the attractiveness function is motivated by the empirical results obtained in [11]. However, in this paper we also consider the attractiveness function

$$\text{attr}_t(i) = q(i)I[i > t - N(n)]$$

which approximates $q(i) e^{-(t-i)/N(n)}$. We do this because both attractiveness functions are similar in terms of the degree distribution, but the theoretical analysis of the attractiveness function $q(i)I[i > t - N(n)]$ is less complicated, therefore it can be considered as the natural first step. In addition, the attractiveness function $q(i)I[i > t - N(n)]$ has its own practical intuition, but unfortunately it cannot model the recency property well (see Section 4.2 for the discussion). In this paper, we analyse both attractiveness functions.

Let us remark that according to the definition of the model the attractiveness of vertices decays rapidly with time. Thus, two vertices with a big age difference are unlikely to be connected and the diameter of a network grows as $n/N(n)$. In this sense, we always observe a chain structure, as it was

noted in [13] for $\alpha > 1$. However, if one considers a subgraph induced by $\sim N(n)$ consecutive vertices, it will be similar to a standard scale-free network. Further we omit n in the notation $N(n)$.

4. Attractiveness function $q(i)I[i > t - N]$

In this section, we assume that the attractiveness function of a vertex i is $\text{attr}_t(i) = q(i)I[i > t - N]$. The indicator function means that a vertex i accumulates incoming edges only during the next N steps after its appearance and we call this period a *lifespan* of a vertex. We say that during this lifespan a vertex is *alive*, after this period a vertex *dies*.

As discussed in Section 2, we also assume that the random variables ζ_1, ζ_2, \dots have the Pareto distribution with the density function $f(x) = \gamma a^\gamma I[x > a]/x^{\gamma+1}$, where $\gamma > 1$, $a > 0$. Further we denote by ζ a random variable with the Pareto distribution defined above.

Finally, our random graph has the following parameters: (1) number of vertices n , (2) vertex out-degree m , (3) lifespan length N , (4) quality exponent γ and (5) minimal quality a .

4.1 Degree distribution

4.1.1 *Results.* In order to simplify the statements of theorems, we introduce an additional constant α . If $\gamma > 2$, then we fix $\alpha = 2$. If $1 < \gamma \leq 2$, then α can be any constant such that $1 < \alpha < \gamma$.

Let $N_n(d)$ be the number of vertices with degree d in G_n . We prove the following theorem.

THEOREM 1 Assume that $d = d(n)$ increases with n and $d = o(\min\{(n/N)^{1/(\gamma+1)}, N^{(\alpha-1)/(\gamma+\alpha+1)}\})$, then

$$\frac{E[N_n(d)]}{n} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma-1)m}{\gamma} \right)^\gamma (1 + o(1)).$$

Theorem 1 shows that the expected number of vertices of degree d decreases as $d^{-\gamma-1}$. In order to get the power-law degree distribution we also need to prove the concentration of the number of vertices with degree d near its expectation.

THEOREM 2 For every d the following inequality holds:

$$P(|N_n(d) - E[N_n(d)]| \geq \sqrt{Nn \log n}) \leq \frac{2}{\log n}.$$

Note that for $d = o((n/N \log n)^{1/2(\gamma+1)})$ we have $\sqrt{Nn \log n} = o(n/d^{\gamma+1})$, so Theorem 2 gives the concentration.

We prove Theorem 1 in Sections 4.1.2 and 4.1.3. Theorem 2 is proved in Section 4.1.4.

4.1.2 *Concentration of the weight.* Let us now fix some n and $N = N(n)$. In this section, we consider only the vertices $N \leq p \leq n - N + 1$.

Let us denote by $\bar{d}(p)$ the degree of a vertex p after its death and by $\bar{d}_{\text{in}}(p)$ the incoming degree of a vertex p after its death, i.e. $\bar{d}_{\text{in}}(p) = \bar{d}(p) - m$. By $Q(t)$, we denote the sum of qualities of the alive

vertices at the t th step, i.e.

$$Q(t) = \sum_{k=t-N}^{t-1} q(k).$$

We also say that $Q(t)$ is the weight of vertices at t th step. Note that

$$E[\bar{d}_{in}(p) \mid q(p-N+1), \dots, q(p+N-1)] = \sum_{i=1}^N \frac{mq(p)}{Q(p+i)}.$$

Indeed, for each $1 \leq i \leq N$ the probability of an edge $(p+i, p)$ is equal to $mq(p)/Q(p+i)$ according to the definition of the model, since $Q(p+i)$ is the overall attractiveness of all vertices at $(p+i)$ th step.

Consider the lifespan of a vertex p with a quality $q(p)$. We have $E[Q(p+i) \mid q(p)] = q(p) + (N-1)E[\zeta]$ for $1 \leq i \leq N$. We want to estimate the probability of this weight $E[Q(p+i) \mid q(p)]$ to deviate from the value $NE[\zeta]$.

Let ξ_1, \dots, ξ_{N-1} be the weights of vertices $p-N+1, \dots, p-1$ and $\eta_1, \dots, \eta_{N-1}$ be the weights of vertices $p+1, \dots, p+N-1$. Let $W_p^q(i)$ be the overall weight of all living vertices when the age of p equals i given that p has the quality q , i.e.

$$W_p^q(i) = \sum_{k=1}^{i-1} \eta_k + q + \sum_{k=i}^{N-1} \xi_k.$$

We will need the following auxiliary lemma.

LEMMA 1 Let ξ_1, \dots, ξ_n be mutually independent random variables, $E[\xi_i] = 0$, $E[|\xi_i|^\alpha] < \infty$, $1 \leq \alpha \leq 2$, then

$$E[|\xi_1 + \dots + \xi_n|^\alpha] \leq 2^\alpha (E[|\xi_1|^\alpha] + \dots + E[|\xi_n|^\alpha]).$$

We placed the proof of this lemma in Appendix.

THEOREM 3 Consider a vertex p with a quality q such that $N \leq p \leq n - N + 1$. If for some constant $c > 0$ we have $|q - E[\zeta]| \leq N^c/2$, then

$$P\left(\max_{1 \leq i \leq N} |W_p^q(i) - NE[\zeta]| \geq N^c\right) = O\left(\frac{E[|\zeta - E[\zeta]|^\alpha]}{N^{\alpha c - 1}}\right).$$

Proof. Note that

$$P\left(\max_{1 \leq i \leq N} |W_p^q(i) - E[W_p^q(1)]| \geq x\right) \leq P\left(|W_p^q(1) - E[W_p^q(1)]| \geq x/2\right) + P\left(\max_{2 \leq i \leq N} |W_p^q(i) - W_p^q(1)| \geq x/2\right).$$

Indeed, $\max_{1 \leq i \leq N} |W_p^q(i) - E[W_p^q(1)]| \leq |W_p^q(1) - E[W_p^q(1)]| + \max_{2 \leq i \leq N} |W_p^q(i) - W_p^q(1)|$ and if $\max_{1 \leq i \leq N} |W_p^q(i) - E[W_p^q(1)]| \geq x$, then either $|W_p^q(1) - E[W_p^q(1)]| \geq x/2$ or $\max_{2 \leq i \leq N} |W_p^q(i) - W_p^q(1)| \geq x/2$.

In the case $\gamma > 2$, the random variables have finite variances and we can apply Chebyshev's and Kolmogorov's inequalities.

Chebyshev's inequality gives

$$\mathbb{P}(|W_p^q(1) - \mathbb{E}[W_p^q(1)]| \geq x/2) \leq \frac{4N \operatorname{Var}[\zeta]}{x^2}.$$

Kolmogorov's inequality gives

$$\mathbb{P}\left(\max_{2 \leq i \leq N} |W_p^q(i) - W_p^q(1)| \geq x/2\right) = \mathbb{P}\left(\max_{1 \leq i \leq N-1} \left|\sum_{k=1}^i (\eta_k - \xi_k)\right| \geq x/2\right) \leq \frac{8N \operatorname{Var}[\zeta]}{x^2}.$$

So, finally we get

$$\mathbb{P}\left(\max_{1 \leq i \leq N} |W_p^q(i) - \mathbb{E}[W_p^q(1)]| \geq x\right) \leq \frac{12N \operatorname{Var}[\zeta]}{x^2}.$$

Take $x = N^c/2$ and note that $|\mathbb{E}[W_p^q(1)] - N\mathbb{E}[\zeta]| = |q - \mathbb{E}[\zeta]| \leq N^c/2$. Therefore we get

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq i \leq N} |W_p^q(i) - N\mathbb{E}[\zeta]| \geq N^c\right) &\leq \mathbb{P}\left(\max_{1 \leq i \leq N} |W_p^q(i) - \mathbb{E}[W_p^q(1)]| \geq N^c/2\right) \\ &\leq \frac{48 \operatorname{Var}[\zeta]}{N^{2c-1}} = O\left(\frac{\mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{N^{\alpha c-1}}\right), \end{aligned}$$

since $\alpha = 2$ in this case.

Now consider the case $1 < \gamma \leq 2$. In this case, we have $1 < \alpha < \gamma$. We cannot use Chebyshev's inequality now, but we can apply Markov's inequality and Lemma 1:

$$\begin{aligned} \mathbb{P}(|W_p^q(1) - \mathbb{E}[W_p^q(1)]| \geq x/2) &= \mathbb{P}(|W_p^q(1) - \mathbb{E}[W_p^q(1)]|^\alpha \geq (x/2)^\alpha) \\ &\leq \frac{\mathbb{E}[|W_p^q(1) - \mathbb{E}[W_p^q(1)]|^\alpha]}{(x/2)^\alpha} \leq \frac{4^\alpha N \mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{x^\alpha}. \end{aligned}$$

Instead of Kolmogorov's inequality, we use Doob's martingale inequality and Lemma 1. Note that $S_i = \left|\sum_{j=1}^i (\eta_j - \xi_j)\right|$ is a submartingale as a convex function of a martingale. Thus,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq i \leq N-1} \left|\sum_{j=1}^i (\eta_j - \xi_j)\right| \geq x/2\right) &\leq \frac{\mathbb{E}\left[\left|\sum_{j=1}^{N-1} (\eta_j - \xi_j)\right|^\alpha\right]}{(x/2)^\alpha} \\ &\leq \frac{4^\alpha N \mathbb{E}[|\eta_1 - \xi_1|^\alpha]}{x^\alpha} \leq \frac{8^\alpha N \mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{x^\alpha}. \end{aligned}$$

So, finally we get

$$\mathbb{P}\left(\max_{1 \leq i \leq N} |W_p^q(i) - \mathbb{E}[W_p^q(1)]| \geq x\right) \leq \frac{4^\alpha (2^\alpha + 1) N \mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{x^\alpha}.$$

Now take $x = N^c/2$ and note that $|\mathbb{E}[W_p^q(1)] - N\mathbb{E}[\zeta]| \leq N^c/2$. As before, we can estimate

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq i \leq N} |W_p^q(i) - N\mathbb{E}[\zeta]| \geq N^c\right) &\leq \frac{8^\alpha (2^\alpha + 1) N \mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{N^{\alpha c}} \\ &\leq \frac{320 \mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{N^{\alpha c - 1}} = O\left(\frac{\mathbb{E}[|\zeta - \mathbb{E}[\zeta]|^\alpha]}{N^{\alpha c - 1}}\right). \end{aligned}$$

This concludes the proof of Theorem 3. □

4.1.3 Expectation. Let $\rho(d, q)$ be the conditional probability that a vertex p such that $N \leq p \leq n - N + 1$ with a quality q has an in-degree d , i.e. $\rho(d, q) = \mathbb{P}(\bar{d}_{\text{in}}(p) = d \mid q(p) = q)$. Note that $\rho(d, q)$ does not depend on p . By $N_n^{\text{in}}(d)$ we denote the number of vertices with in-degree d , so $N_n^{\text{in}}(d) = N_n(d + m)$. The expectation of $N_n^{\text{in}}(d)$ is

$$\mathbb{E}[N_n^{\text{in}}(d)] = (n - 2N) \int_a^\infty f(q) \rho(d, q) dq + r(N), \tag{1}$$

where $f(q)$ is the density function of Pareto distribution defined above and $r(N)$, $0 \leq r(N) \leq 2N$, is the error term. We have this error term since the first and the last N vertices behave differently.

Let c be some positive constant. We estimate the integral

$$I = \int_a^\infty f(q) \rho(d, q) dq = \int_a^{N^c/2} f(q) \rho(d, q) dq + \int_{N^c/2}^\infty f(q) \rho(d, q) dq = I_1 + I_2.$$

Note that

$$I_2 = \int_{N^c/2}^\infty f(q) \rho(d, q) dq \leq \int_{N^c/2}^\infty f(q) dq = \int_{N^c/2}^\infty \frac{\gamma a^\gamma}{q^{\gamma+1}} dq = \frac{(2a)^\gamma}{N^{c\gamma}}. \tag{2}$$

Consider the event

$$A = \left\{ \max_{1 \leq i \leq N} |Q(p + i) - N\mathbb{E}[\zeta]| \leq N^c \right\}$$

and the following conditional probabilities:

$$\begin{aligned} \rho_A(d, q) &= \mathbb{P}(\bar{d}_{\text{in}}(p) = d \mid q(p) = q, A), \\ \rho_{\bar{A}}(d, q) &= \mathbb{P}(\bar{d}_{\text{in}}(p) = d \mid q(p) = q, \bar{A}). \end{aligned}$$

Then we have

$$\rho(d, q) = \rho_A(d, q) \mathbb{P}(A \mid q(p) = q) + \rho_{\bar{A}}(d, q) \mathbb{P}(\bar{A} \mid q(p) = q). \tag{3}$$

Let us use this representation to split I_1 into two integrals using (3):

$$I_1 = \int_a^{N^c/2} f(q) \rho_A(d, q) \mathbb{P}(A_q) dq + \int_a^{N^c/2} f(q) \rho_{\bar{A}}(d, q) \mathbb{P}(\bar{A}_q) dq = I_1^1 + I_1^2,$$

where we use the following notation:

$$A_q = [A \mid q(p) = q] = \left\{ \max_{1 \leq i \leq N} |W_p^q(i) - NE[\zeta]| \leq N^c \right\},$$

$$\bar{A}_q = [\bar{A} \mid q(p) = q] = \left\{ \max_{1 \leq i \leq N} |W_p^q(i) - NE[\zeta]| > N^c \right\}.$$

Let us assume that $N^c/2 > E[\zeta]$, this holds if N is large enough (the fact that N grows follows from the statement of Theorem 1, while $E[\zeta]$ is constant). Note that

$$I_1^2 \leq \max_{q \leq N^c/2} P(\bar{A}_q) \tag{4}$$

and since $q \leq N^c/2$ Theorem 3 gives us an upper bound for it, i.e.

$$\max_{q \leq N^c/2} P(\bar{A}_q) = O(N^{1-\alpha c}), \tag{5}$$

where $\alpha = 2$ for $\gamma > 2$ and $1 < \alpha < \gamma$ for $1 < \gamma \leq 2$.

So, let us now focus on I_1^1 . First we estimate $\rho_A(d, q)$. Recall that $\rho_A(d, q) = P(\bar{d}_{in}(p) = d \mid q(p) = q, A)$. Note that during the lifespan of a vertex p there are mN mutually independent edges which may lead to p . For an edge from a vertex $p+i$ the probability to choose p is $q/W_p^q(i)$. Given the event A_q we have $NE[\zeta] - N^c \leq W_p^q(i) \leq NE[\zeta] + N^c$. Therefore, we have the following bounds for $\rho_A(q, d)$:

$$\begin{aligned} \binom{mN}{d} \left(\frac{q}{NE[\zeta] + N^c} \right)^d \left(1 - \frac{q}{NE[\zeta] - N^c} \right)^{mN-d} \\ \leq \rho_A(d, q) \leq \binom{mN}{d} \left(\frac{q}{NE[\zeta] - N^c} \right)^d \left(1 - \frac{d}{NE[\zeta] + N^c} \right)^{mN-d}. \end{aligned}$$

Thus, $(1 - \max_{q \leq N^c/2} P(\bar{A}_q))S_- \leq I_1^1 \leq S_+$ where

$$S_{\mp} = \int_a^{N^c/2} f(q) \binom{mN}{d} \left(\frac{q}{NE[\zeta] \pm N^c} \right)^d \left(1 - \frac{q}{NE[\zeta] \mp N^c} \right)^{mN-d} dq.$$

We will use the following lemma.

LEMMA 2 Assume that both N and d grow but $d = o(N^{1-c})$. If $1/2 \leq c \leq 1$, then

$$S_{\mp} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma-1)m}{\gamma} \right)^{\gamma} (1 + o(1)).$$

We placed the proof of this technical lemma in Appendix. Now we will use this lemma to prove the theorem. Using Equations (1), (2) and (4) we get the following bounds for $E[N_n(d + m)]/n$:

$$\begin{aligned} & \left(1 - \frac{2N}{n}\right) \underbrace{\left(1 - \max_{q \leq N^c/2} P(\bar{A}_q)\right)}_{\leq I_1^1} S_- \\ & \leq \frac{E[N_n(d + m)]}{n} \leq \underbrace{S_+}_{\geq I_1^1} + \underbrace{\max_{q \leq N^c/2} P(\bar{A}_q)}_{\geq I_1^2} + \underbrace{\frac{(2a)^\gamma}{N^{c\gamma}}}_{\geq I_2} + \frac{2N}{n}. \end{aligned} \tag{6}$$

Now we show that for some parameter c all error terms in Equation (6) are negligible in comparison with the main term $d^{-\gamma-1}$ from Lemma 2. We take $c = (\gamma + 2)/(\gamma + \alpha + 1)$. Note that we can apply Lemma 2 since $d = o(N^{1-c})$ due to the statement of Theorem 1.

1. $N/n = o(d^{-\gamma-1})$, since $d = o((n/N)^{1/(\gamma+1)})$.
2. $(2a)^\gamma/N^{c\gamma} = o(d^{-\gamma-1})$ if $d = o(N^{c\gamma/(\gamma+1)})$. This holds for $c = (\gamma + 2)/(\gamma + \alpha + 1)$ and $d = o(N^{(\alpha-1)/(\gamma+\alpha+1)})$.
3. For $\gamma \leq 2$, $P(\bar{A}_q) = O(N^{1-c\alpha}) = o(d^{-\gamma-1})$ if $d = o(N^{(c\alpha-1)/(\gamma+1)})$, i.e. $d = o(N^{(\alpha-1)/(\gamma+\alpha+1)})$. Here we used Equation (5).

We demonstrated that all the error terms in Equation (6) equal $o(d^{-\gamma-1})$. Therefore, from Lemma 2 we get

$$\frac{E[N_n(d + m)]}{n} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma - 1)m}{\gamma}\right)^\gamma (1 + o(1)).$$

To conclude the proof of Theorem 1 it remains to note that the asymptotic for $N_n(d)$ is the same as for $N_n(d + m)$.

4.1.4 *Concentration.* We use Chebyshev’s inequality to prove concentration. In order to do this we first estimate $\text{Var}[N_n(d)]$. Note that if $|i - j| \geq N$, then the degrees of i and j are independent. Therefore

$$\text{Var}[N_n(d)] = \sum_{i,j=1}^n (P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d)P(d_n(j) = d)) \leq 2nN.$$

Applying Chebyshev’s inequality we get

$$P\left(|N_n(d) - E[N_n(d)]| > \sqrt{Nn \log n}\right) \leq \frac{\text{Var}[N_n(d)]}{Nn \log n} \leq \frac{2}{\log n}.$$

REMARK Note that instead we could use Azuma–Hoeffding inequality, since $|E[N_n(d)|G_i] - E[N_n(d)|G_{i-1}]| \leq (N + 1)m$. In this case we get

$$P(|N_n(d) - E[N_n(d)]| \geq \sqrt{n \log n}(N + 1)) \leq 2n^{-1/2m^2}.$$

So, on the one hand the range of degrees for which we get concentration is smaller in this case. We get concentration for $d = o((\sqrt{n}/N\sqrt{\log n})^{1/(\gamma+1)})$. On the other hand, the concentration is tighter, so we can say that for all d in this range the number of vertices of degree d is near its expectation.

4.2 Recency property

Let $e(T)$ be the fraction of edges in a graph which connect vertices with age difference greater than T , i.e. vertices i and j with $|i - j| > T$. In [11], a *recency property* was introduced, which reflects the fact that new media pages tend to connect to other media pages of similar age. Namely, for the media related part of the Web it was shown that $e(T)$ decreases exponentially fast. In this section, we show that we have linear decay of $e(T)$ for the model under consideration.

THEOREM 4 For any integer T

$$E[e(T)] = \begin{cases} 1 - \frac{T}{N} + O\left(\frac{N}{n}\right), & \text{if } T \leq N; \\ 0, & \text{if } T > N. \end{cases}$$

Proof. Consider any vertex $n > N$ and any edge ni , $i < n$, drawn from this vertex. The probability that $n - i > T$ is the probability to choose one vertex from $n - N, \dots, n - T - 1$. Since qualities of vertices are i.i.d. random variables, this probability equals $(N - T)/N$. From this the theorem follows. \square

THEOREM 5

$$P\left(|e(T) - E[e(T)]| \geq \sqrt{\frac{N \log n}{n}}\right) \leq \frac{2}{\log n}.$$

Proof. Here we again use Chebyshev’s inequality. Let e_i and e_j be any two different edges drawn from vertices i and j . For an edge e let $l(e)$ be the age difference between the endpoints of the edge.

Note that if $|i - j| \geq N$, then $l(e_i)$ and $l(e_j)$ are independent:

$$P(l(e_i) > T, l(e_j) > T) - P(l(e_i) > T)P(l(e_j) > T) = 0.$$

From this we get $\text{Var}[mne(T)] \leq 2m^2Nn$, since we take into account only the summands corresponding to edges e_i and e_j with $|i - j| \leq N$ and $P(l(e_i) > T, l(e_j) > T) - P(l(e_i) > T)P(l(e_j) > T) \leq 1$.

Therefore,

$$P\left(mn |e(T) - E[e(T)]| \geq m\sqrt{Nn \log n}\right) \leq \frac{\text{Var}[mne(T)]}{m^2Nn \log n} \leq \frac{2}{\log n}. \quad \square$$

Finally, let us discuss an intuition behind the recency factor $I[i > t - N]$. This factor has the following natural interpretation. Links to a lot of media pages can usually be found on some pages which are content sources. And new pages are popular while they can be found on such content sources. After some period of time other new pages appear on a content source and they replace old ones. Therefore, it seems natural to assume that after some period of time old pages become unpopular. However, as it turned out, this recency factor can model only linear decay of $e(T)$, while we want to see exponential one. One possible reason for this lack of agreement is that the most recent links are usually placed near the top of a page and they may attract more attention because of this.

5. Attractiveness function $q(i) e^{-(t-i)/N}$

Now we switch to the attractiveness function $q(i) e^{-(t-i)/N}$. In this case, the popularity of a vertex decreases exponentially with the age of the vertex. Again, we assume that the random variables ζ_1, ζ_2, \dots have the Pareto distribution with the density function $f(x) = \gamma a^\gamma I[x > a]/x^{\gamma+1}$, where $\gamma > 1, a > 0$. And ζ again is a random variable with the Pareto distribution defined above.

5.1 Degree distribution

5.1.1 Results. For the model with the exponential recency factor we get the results similar to ones for the model with the indicator recency factor (see Section 4.1.1).

Recall that a constant α is defined as follows: if $\gamma > 2$, then $\alpha = 2$; if $1 < \gamma \leq 2$, then α can be any constant such that $1 < \alpha < \gamma$.

THEOREM 6 If $d = d(n)$ increases with n and $d = o(\min\{(n/N \log N)^{1/(\gamma+1)}, N^{(\alpha-1)/(\alpha+(\gamma+1)(\alpha+1))}\})$, then

$$\frac{E[N_n(d)]}{n} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma - 1)m}{\gamma} \right)^\gamma (1 + o(1)).$$

Again, the expectation of the number of vertices with degree d decreases as $d^{-\gamma-1}$. The next theorem shows that the number of vertices of degree d is concentrated near its expectation.

THEOREM 7 For every d the following inequality holds:

$$P\left(|N_n(d) - E[N_n(d)]| > \sqrt{Nn \log n}\right) = O\left(\frac{1}{\log n}\right).$$

As before, for $d = o((n/N \log n)^{1/2(\gamma+1)})$ we have $\sqrt{Nn \log n} = o(E[N_n(d)])$ and Theorem 7 gives the concentration.

We prove Theorem 6 in Sections 5.1.2 and 5.1.3. Theorem 7 is proved in Section 5.1.4.

5.1.2 Concentration of the overall attractiveness. We fix some n and $N = N(n)$.

By $Q(t)$ we denote the total attractiveness of all vertices at t th step, i.e.

$$Q(t) = \sum_{k=1}^{t-1} q(k) e^{-(t-k-1)/N}.$$

The average value of $Q(t)$ is

$$E[Q(t)] = E[\zeta] \sum_{k=0}^{t-2} e^{-k/N} = E[\zeta] \frac{1 - e^{-(t-1)/N}}{1 - e^{-1/N}} = NE[\zeta] (1 + O(e^{-t/N}) + O(1/N)).$$

If $t > N \log N$, then

$$E[Q(t)] = NE[\zeta](1 + O(1/N)).$$

Again, by $W_p^q(i)$ we denote the total attractiveness of all vertices when the age of p equals i given the quality q of the vertex p .

THEOREM 8 Fix some positive constant c . Let $\varphi(N)$ be any function such that $\varphi(N) > \log(CN)$ for some $C > 0$. Then for any $p > N\varphi(N)$ with a quality q satisfying $|q - E[\zeta]| \leq N^c/3$ we have

$$P\left(\max_{1 \leq i \leq N\varphi(N)} |W_p^q(i) - NE[\zeta]| \geq N^c\right) = O\left(e^{\alpha\varphi(N)} N^{1-\alpha c}\right).$$

Proof. Note that $E[Q(p+i+1) | Q(p+i)] = Q(p+i)e^{-1/N} + E[\zeta]$. Therefore $X_i = e^{i/N} \cdot (Q(p+i) - E[\zeta]/(1 - e^{-1/N}))$ is a martingale. Indeed,

$$E[X_{i+1} | X_i] = (Q(p+i)e^{-1/N} + E[\zeta])e^{(i+1)/N} - \frac{e^{(i+1)/N}E[\zeta]}{1 - e^{-1/N}} = X_i.$$

So, we can apply Doob's inequality for a submartingale $|X_i|$:

$$\begin{aligned} P\left(\max_{1 \leq i \leq N\varphi(N)} \left|e^{i/N} \left(Q(p+i) - \frac{E[\zeta]}{1 - e^{-1/N}}\right)\right| \geq x\right) \\ \leq \frac{E\left[\left|e^{N\varphi(N)/N} (Q(p+N\varphi(N)) - E[\zeta]/(1 - e^{-1/N}))\right|^\alpha\right]}{x^\alpha}. \end{aligned}$$

So, we get

$$\begin{aligned} P\left(\max_{1 \leq i \leq N\varphi(N)} \left|Q(p+i) - \frac{E[\zeta]}{1 - e^{-1/N}}\right| \geq N^c/3\right) \\ \leq \frac{3^\alpha e^{\alpha\varphi(N)} E\left[\left|Q(p+N\varphi(N)) - E[\zeta]/(1 - e^{-1/N})\right|^\alpha\right]}{N^{\alpha c}}. \end{aligned}$$

First, consider the case $\gamma > 2$. In this case we have $\alpha = 2$. Using

$$\frac{E[\zeta]}{1 - e^{-1/N}} = E[Q(p+N\varphi(N))] + E[\zeta] \sum_{k=p+N\varphi(N)-1}^{\infty} e^{-k/N}$$

we get

$$\begin{aligned} E\left[\left(Q(p+N\varphi(N)) - \frac{E\zeta}{1 - e^{-1/N}}\right)^2\right] \\ = E\left[(Q(p+N\varphi(N)) - EQ(p+N\varphi(N)))^2\right] + (E[\zeta])^2 \left(\frac{e^{-(p+N\varphi(N)-1)/N}}{1 - e^{-1/N}}\right)^2 \\ = \text{Var}[\zeta] \sum_{k=0}^{p+N\varphi(N)-2} e^{-2k/N} + O\left(\frac{e^{-2(N \log(CN) + N\varphi(N))/N}}{(1 - e^{-1/N})^2}\right) \\ = O\left(\frac{1}{1 - e^{-2/N}} + e^{-2\varphi(N)}\right) = O(N). \end{aligned}$$

So,

$$P\left(\max_{1 \leq i \leq N\varphi(N)} \left|Q(p+i) - \frac{E[\zeta]}{1 - e^{-1/N}}\right| \geq N^c/3\right) = O\left(e^{2\varphi(N)} N^{1-2c}\right).$$

Now we can estimate $W_p^q(i)$ which is $Q(p+i)$ given the quality q of the vertex p . We have $|q - E[\xi]| \leq N^c/3$ and $|E[\xi]/(1 - e^{-1/N}) - NE[\xi]| \leq N^c/3$ for large N , therefore

$$P\left(\max_{1 \leq i \leq N\varphi(N)} |W_p^q(i) - NE[\xi]| \geq N^c\right) = O(e^{2\varphi(N)} N^{1-2c}).$$

Similarly, for $\gamma \leq 2$ using Lemma 1 we get

$$P\left(\max_{1 \leq i \leq N\varphi(N)} |W_p^q(i) - NE[\xi]| \geq N^c\right) = O(e^{\alpha\varphi(N)} N^{1-\alpha c}). \quad \square$$

5.1.3 Expectation. Let $\varphi(N)$ be any function such that $\varphi(N) > \log(CN)$ for some $C > 0$. Let $\rho(d, q)$ be the conditional probability that a vertex p such that $N\varphi(N) \leq p \leq n - N\varphi(N) + 1$ has an in-degree d given a quality q of this vertex, i.e. $\rho(d, q) = P(\bar{d}_{\text{in}}(p) = d | q(p) = q)$. We omit n and p in the notation $\rho(d, q)$ because, as we will see, we get similar bounds for $\rho(d, q)$ for all p such that $N\varphi(N) \leq p \leq n - N\varphi(N) + 1$. Using this notation, we get the following equality:

$$E[N_n^{\text{in}}(d)] = (n - 2N\varphi(N)) \int_a^\infty f(q)\rho(d, q) dq + r(N), \quad (7)$$

where $f(q)$ is the density function of Pareto distribution and $r(N)$, $0 \leq r(N) \leq 2N\varphi(N)$ is the error term.

Let r and c be some constants such that $0 < r < 1/2$ and $1/2 < c < 1$. As in Section 4.1.3, we split the integral

$$I = \int_a^\infty f(q)\rho(d, q) dq = \int_a^{N^r} f(q)\rho(d, q) dq + \int_{N^r}^\infty f(q)\rho(d, q) dq = I_1 + I_2$$

and

$$I_2 \leq \int_{N^r}^\infty f(q) dq = \frac{a^r}{N^{r\gamma}}. \quad (8)$$

The event A is defined as in Section 4.1.3:

$$A = \left\{ \max_{1 \leq i \leq N\varphi(N)} |Q(p+i) - NE[\xi]| \leq N^c \right\}.$$

We again split I_1 into two integrals:

$$I_1 = \int_a^{N^r} f(q)\rho_A(d, q)P(A_q) dq + \int_a^{N^r} f(q)\rho_{\bar{A}}(d, q)P(\bar{A}_q) dq = I_1^1 + I_1^2,$$

where

$$A_q = [A | q(p) = q] = \left\{ \max_{1 \leq i \leq N\varphi(N)} |W_p^q(i) - NE[\xi]| \leq N^c \right\},$$

$$\bar{A}_q = [\bar{A} | q(p) = q] = \left\{ \max_{1 \leq i \leq N\varphi(N)} |W_p^q(i) - NE[\xi]| > N^c \right\}.$$

We can estimate

$$I_1^2 \leq \max_{q \leq N^r} P(\bar{A}_q) \tag{9}$$

and for $q \leq N^r$ Theorem 8 gives the upper bound for $P(\bar{A}_q)$ (since $N^r < N^c/3$ and $|q - E[\zeta]| \leq N^r$ for large N):

$$\max_{q \leq N^r} P(\bar{A}_q) = O(e^{\alpha\varphi(N)} N^{1-\alpha c}). \tag{10}$$

Let us now focus on I_1^1 . Consider an event $R_p^q(k)$ that there is an edge from at least one vertex $p + i$ with $i \geq k$ to a vertex p with a quality q . Then for $k > N$ conditional probability of $R_p^q(k)$ given A_q can be estimated as follows:

$$\begin{aligned} P(R_p^q(k) | A_q) &\leq \sum_{i=k}^{\infty} P(\text{edge } (p+i, p) \text{ belongs to } G_n | A_q) \\ &\leq \sum_{i=k}^{\infty} \frac{mq e^{-(i-1)/N}}{\sum_{j=0}^N a e^{-j}/N} \leq \sum_{i=k}^{\infty} \frac{mq e^{-(i-1)/N}}{aN/2} = O(q e^{-k/N}). \end{aligned}$$

This estimate means that the most contribution to the final degree of a vertex is made during the first several steps after its appearance and we have the following bounds for $\rho_A(d, q)$:

$$\rho_A(d, q) = \rho_{\mp}(d, q) + O(q e^{-\varphi(N)}),$$

where $\rho_{\mp}(d, q)$ are lower and upper bounds for the probability that a vertex p with a quality q has an in-degree d in $\tilde{G}_{p+N\varphi(N)}^n$ given A_q . We can estimate $\rho_{\mp}(d, q)$ in the following way. A vertex p has an in-degree d in $\tilde{G}_{p+N\varphi(N)}^n$ if d edges out of $m\varphi(N)N$ are connected to this vertex and others are not. For every set of indexes $0 \leq i_1 < \dots < i_d \leq m\varphi(N)N$ we should multiply the probabilities that the corresponding edges go to the vertex p . Given A_q , these probabilities can be estimated by $q e^{-[i_j/m]/N} / (NE[\zeta] \mp N^c)$. And we should also multiply the obtained product by the probabilities that other edges are not connected to p , i.e. $(1 - q e^{-[i/m]/N} / (NE[\zeta] \pm N^c))$ for the corresponding indexes i . Finally, we obtain

$$\rho_{\mp}(d, q) = \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{q e^{-[i/m]/N}}{NE[\zeta] \pm N^c} \right) \sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} \prod_{j=1}^d \frac{q e^{-[i_j/m]/N} / (NE[\zeta] \mp N^c)}{1 - q e^{-[i_j/m]/N} / (NE[\zeta] \pm N^c)}.$$

Now we put

$$S_{\mp}(d, q) := \int_a^{N^r} f(q) \rho_{\mp}(d, q) dq.$$

Using this notation, we can estimate I_1^1 in the following way:

$$I_1^1 \leq \int_a^{N^r} f(q) \rho_A(d, q) dq \leq S_+ + O\left(\int_a^{\infty} f(q) q e^{-\varphi(N)} dq\right), \tag{11}$$

$$I_1^1 \geq \left(1 - \max_{q \leq N^r} P(\bar{A}_q)\right) S_- + O\left(\int_a^{\infty} f(q) q e^{-\varphi(N)} dq\right). \tag{12}$$

We estimate S_{\mp} in the following way.

LEMMA 3 Assume that both d and N grow, $d = o(N^{1-c})$, $d = o(e^{\varphi(N)})$ and $q \leq N^r$, then

$$S_{\mp}(d, q) = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma - 1)m}{\gamma} \right)^{\gamma} (1 + o(1)).$$

We placed the proof of this technical lemma in the appendix. Finally, using Equations (7–9), (11) and (12), we get

$$\begin{aligned} & \left(1 - \frac{2N\varphi(N)}{n} \right) \underbrace{\left(1 - \max_{q \leq N^r} P(\bar{A}_q) \right) S_- + O \left(\int_a^{\infty} f(q)q e^{-\varphi(N)} dq \right)}_{\leq I_1^!} \\ & \leq \frac{E[N_n(d + m)]}{n} \leq \underbrace{S_+ + O \left(\int_a^{\infty} f(q)q e^{-\varphi(N)} dq \right)}_{\geq I_1^!} \\ & + \underbrace{\max_{q \leq N^r} P(\bar{A}_q)}_{\geq I_2^!} + \underbrace{\frac{a^{\gamma}}{N^{r\gamma}}}_{\geq I_2} + \frac{2N\varphi(N)}{n}. \end{aligned} \tag{13}$$

We want all the error terms in Equation (13) to be $o(d^{-\gamma-1})$. In order to do this, we need to find the proper values of c and $\varphi(n)$. Note that we have already assumed that $d = o(N^{1-c})$ and $d = o(e^{\varphi(N)})$. We have to show that the following conditions hold.

1. $N\varphi(N)/n = o(d^{-\gamma-1})$ if $d = o((n/N\varphi(N))^{1/(\gamma+1)})$. This holds under the conditions of the theorem.
2. $a^{\gamma}/N^{r\gamma} = o(d^{-\gamma-1})$ if $d = o(N^{r\gamma/(\gamma+1)})$. Put $r = 9/22$, then we have $d = o(N^{r\gamma/(\gamma+1)})$ under the conditions of the theorem since $N^{9\gamma/22(\gamma+1)} \geq N^{(\alpha-1)/(\alpha+(\gamma+1)(\alpha+1))}$ for both $\gamma > 2$ and $1 \leq \gamma \leq 2$.
3. $\max_{q \leq N^r} P(\bar{A}_q) = O(e^{\alpha\varphi(N)}N^{1-\alpha c}) = o(d^{-\gamma-1})$ if $e^{\alpha\varphi(N)} = o(N^{\alpha c-1}d^{-\gamma-1})$. Here we used Equation (10).
4. $\int_a^{\infty} f(q)O(q e^{-\varphi(N)}) dq = O(e^{-\varphi(N)}) = o(d^{-\gamma-1})$ if $d^{\gamma+1} = o(e^{\varphi(N)})$.

We take $c = (1 + (\gamma + 1)(\alpha + 1))/(\alpha + (\gamma + 1)(\alpha + 1))$ and $\varphi(N) = \log(N(\alpha - 1)(\gamma + 1)/(\alpha + (\gamma + 1)(\alpha + 1)))$. Then for $d = o(N^{(\alpha-1)/(\alpha+(\gamma+1)(\alpha+1))})$ all the above conditions hold. This means that all the error terms in Equation (13) equal $o(d^{-\gamma-1})$. Therefore, we obtained the required asymptotic for $E[N_n(d + m)]/n$.

To conclude the proof of Theorem 6 it remains to note that the asymptotic for $N_n(d)$ is the same as for $N_n(d + m)$.

5.1.4 *Concentration.* We prove Theorem 7 using Chebyshev’s inequality. In order to apply this inequality we first estimate $\text{Var}[N_n(d)]$:

$$\text{Var}[N_n(d)] = \sum_{i,j=1}^n (P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d)P(d_n(j) = d)).$$

Let us estimate the difference $P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d)P(d_n(j) = d)$ for $i < j$. Note that

$$P(d_j(i) = d, d_n(j) = d) = P(d_j(i) = d)P(d_n(j) = d). \tag{14}$$

In order to prove this we first show that (14) holds given all the qualities q_1, \dots, q_n and then integrate over all qualities. Given the qualities, $P(d_j(i) = d, d_n(j) = d)$ is the sum over all $mi < i_1 < \dots < i_d \leq mj$, $mj < j_1 < \dots < j_d \leq mn$ of the probabilities that the corresponding edges $([i_k/m], i)$ and $([j_k/m], j)$ are drawn and all other edges (i', i) with $i < i' \leq j$ and (j', j) with $j < j' \leq n$ are absent. Since qualities are fixed, these events are independent and $P(d_j(i) = d, d_n(j) = d) = P(d_j(i) = d)P(d_n(j) = d)$.

Let $R_p(k)$ be the event that there is an edge from at least one vertex $p + i$ with $i \geq k$ to a vertex p . Then

$$\begin{aligned} & P(d_n(i) = d, d_n(j) = d) - P(d_n(i) = d)P(d_n(j) = d) \\ & \leq P(d_j(i) = d, d_n(j) = d) + P(R_i(j - i)) - P(d_n(i) = d)P(d_n(j) = d) \\ & = P(d_j(i) = d)P(d_n(j) = d) + P(R_i(j - i)) - P(d_n(i) = d)P(d_n(j) = d) \\ & \leq P(d_j(i) = d)P(d_n(j) = d) + P(R_i(j - i))P(d_n(j) = d) \\ & \quad + P(R_i(j - i)) - P(d_n(i) = d)P(d_n(j) = d) \\ & \leq 2P(R_i(j - i)) = 2 \int_a^\infty R_i^q(j - i)f(q) \, dq \\ & = O\left(\int_a^\infty q^{-\gamma-1} e^{-(j-i)/N} \, dq\right) = O\left(e^{-(j-i)/N}\right). \end{aligned}$$

Finally,

$$\text{Var}[N_n(d)] = O\left(\sum_{1 \leq i \leq j \leq n} e^{-(j-i)/N}\right) = O(Nn).$$

Applying Chebyshev's inequality we get

$$P(|N_n(d) - E[N_n(d)]| > \sqrt{Nn \log n}) = O\left(\frac{\text{Var}[N_n(d)]}{Nn \log n}\right) = O\left(\frac{1}{\log n}\right).$$

5.2 Recency property

In this section, we show that the behaviour of $e(T)$ for the model with exponential popularity decay is realistic. It was shown in [11] that $e(T)$ decreases exponentially with T in real data.

First, we compute the expectation of $e(T)$. The following theorem holds.

THEOREM 9 For any integer T

$$E[e(T)] = e^{-T/N} + O\left(\frac{N}{n}\right).$$

Indeed, the probability that an edge from a vertex k goes to a vertex i with $k - i > T$ equals $e^{-T/N} + O(e^{-k/N})$. From this Theorem 9 follows.

As in Section 4.2, we can use Chebyshev's inequality to prove the concentration.

THEOREM 10 For any integer T

$$P\left(|e(T) - E[e(T)]| \geq \sqrt{\frac{N \log n}{n}}\right) = O\left(\frac{1}{\log n}\right).$$

Proof. As before, by e_i and e_j we denote any two different edges drawn from vertices i and j , and $l(e)$ is the age difference between the endpoints of an edge e .

Note that for $i \leq j$

$$P(l(e_i) > T, l(e_j) > T \mid l(e_j) \leq j - i) = P(l(e_i) > T)P(l(e_j) > T \mid l(e_j) \leq j - i).$$

Therefore, we can estimate the following difference:

$$\begin{aligned} & P(l(e_i) > T, l(e_j) > T) - P(l(e_i) > T)P(l(e_j) > T) \\ &= P(l(e_j) \leq j - i)(P(l(e_i) > T, l(e_j) > T \mid l(e_j) \leq j - i) \\ &\quad - P(l(e_i) > T)P(l(e_j) > T \mid l(e_j) \leq j - i)) \\ &\quad + P(l(e_j) > j - i)(P(l(e_i) > T, l(e_j) > T \mid l(e_j) > j - i) \\ &\quad - P(l(e_i) > T)P(l(e_j) > T \mid l(e_j) > j - i)) \leq P(l(e_j) > j - i) \leq e^{-(j-i)/N}. \end{aligned}$$

Thus,

$$\text{Var}[mn e(T)] = O\left(\sum_{1 \leq i < j \leq n} e^{-(j-i)/N}\right) = O(Nn).$$

Finally,

$$P\left(mn |e(T) - E[e(T)]| \geq m\sqrt{Nn \log n}\right) \leq \frac{\text{Var}[mn e(T)]}{m^2 Nn \log n} = O\left(\frac{1}{\log n}\right). \quad \square$$

Theorems 9 and 10 mean that $e(T)$ decays exponentially, as it was observed in real data.

6. Conclusion

In this paper, we analyse recency-based models. The idea of adding the recency factor to the attractiveness function was first suggested in [11]. In this paper we consider the most realistic model proposed in [11] and conduct a rigorous analysis of its properties. In order to do this, we first provide a new formal definition of the model. Then, we justify the fact that the power-law distribution of inherent quality leads to the power-law degree distribution. We also rigorously analyse the recency property, i.e. the behaviour of $e(T)$, and prove that $e(T)$ decreases exponentially as it is observed in some real-world networks.

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Appendix

Proof of Lemma 1

LEMMA 1 Let ξ_1, \dots, ξ_n be mutually independent random variables, $E[\xi_i] = 0$, $E[|\xi_i|^\alpha] < \infty$, $1 \leq \alpha \leq 2$, then

$$E[|\xi_1 + \dots + \xi_n|^\alpha] \leq 2^\alpha (E[|\xi_1|^\alpha] + \dots + E[|\xi_n|^\alpha]).$$

Proof. We use the following two facts.

FACT 1 If ξ and η are independent random variables and η is symmetrically distributed, then for any $1 \leq \alpha \leq 2$

$$E[|\xi + \eta|^\alpha] \leq E[|\xi|^\alpha] + E[|\eta|^\alpha].$$

Proof.

$$E[|\xi + \eta|^\alpha] = \frac{1}{2} (E[|\xi + \eta|^\alpha] + E[|\xi - \eta|^\alpha])$$

and it remains to show that for any x, y and $1 \leq \alpha \leq 2$ we have

$$\frac{1}{2} (|x + y|^\alpha + |x - y|^\alpha) \leq |x|^\alpha + |y|^\alpha.$$

Without loss of generality we assume that $x \geq y \geq 0$ and consider the function $f(x, y) = \frac{1}{2}((x + y)^\alpha + (x - y)^\alpha) - x^\alpha - y^\alpha$. In order to show that $f(x, y) \leq 0$ we note that $f(x, 0) = 0$ and $\partial f(x, y)/\partial y \leq 0$. In turn, $\partial f(x, y)/\partial y \leq 0$ since $\partial f(x, y)/\partial y|_{x=y} \leq 0$ and $\partial^2 f(x, y)/\partial x \partial y \leq 0$. \square

FACT 2 If $\alpha \geq 1$, ξ and η are independent random variables, $E[\eta] = 0$, $E[|\xi|^\alpha] < \infty$, $E[|\eta|^\alpha] < \infty$, then

$$E[|\xi + \eta|^\alpha] \geq E[|\xi|^\alpha].$$

Proof. Fact 2 follows directly from Jensen's inequality. □

Now, let us prove Lemma 1. Consider random variables ξ'_1, \dots, ξ'_n , such that ξ'_i has the same distribution as ξ_i and $\xi_1, \dots, \xi_n, \xi'_1, \dots, \xi'_n$ are mutually independent. Note that $\xi_i - \xi'_i$ is symmetrically distributed for any i . Then from Facts 1 and 2 it follows that

$$E[|\xi_1 + \dots + \xi_n|^\alpha] \leq E[|\xi_1 - \xi'_1 + \dots + \xi_n - \xi'_n|^\alpha] \leq E[|\xi_1 - \xi'_1|^\alpha + \dots + |\xi_n - \xi'_n|^\alpha].$$

Finally, it remains to note that $E[|\xi_i - \xi'_i|^\alpha] \leq 2^\alpha E[|\xi_i|^\alpha]$. □

Proof of Lemma 2

First, recall the statement of Lemma 2 from Section 4.1.3.

LEMMA 2 Assume that both N and d grow but $d = o(N^{1-c})$. If $1/2 \leq c \leq 1$, then

$$S_{\mp} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma-1)m}{\gamma} \right)^\gamma (1 + o(1)).$$

Proof. Recall that

$$S_{\mp} = \int_a^{N^{c/2}} f(q) \binom{mN}{d} \left(\frac{q}{NE[\zeta] \pm N^c} \right)^d \left(1 - \frac{q}{NE[\zeta] \mp N^c} \right)^{mN-d} dq.$$

Let us rewrite S_{\mp} using the incomplete beta-function $B(x; a, b)$

$$\begin{aligned} S_{\mp} &= \int_a^{N^{c/2}} \frac{\gamma a^\gamma}{q^{\gamma+1}} \binom{mN}{d} \left(\frac{q}{NE[\zeta] \pm N^c} \right)^d \left(1 - \frac{q}{NE[\zeta] \mp N^c} \right)^{mN-d} dq \\ &= \frac{\gamma a^\gamma \binom{mN}{d} (NE[\zeta] \mp N^c)^{d-\gamma-1}}{(NE[\zeta] \pm N^c)^d} \\ &\quad \times \int_a^{N^{c/2}} \left(\frac{q}{NE[\zeta] \mp N^c} \right)^{d-\gamma-1} \left(1 - \frac{q}{NE[\zeta] \mp N^c} \right)^{mN-d} dq \\ &= \frac{\gamma a^\gamma \binom{mN}{d} (NE[\zeta] \mp N^c)^{d-\gamma}}{(NE[\zeta] \pm N^c)^d} \cdot \int_{a/(NE[\zeta] \mp N^c)}^{(N^{c/2})/(NE[\zeta] \mp N^c)} x^{d-\gamma-1} (1-x)^{mN-d} dx \end{aligned}$$

$$= \frac{\gamma a^\gamma \binom{mN}{d} (NE[\zeta] \mp N^c)^{d-\gamma}}{(NE[\zeta] \pm N^c)^d} \left(B\left(\frac{N^c/2}{NE[\zeta] \mp N^c}; d - \gamma, mN - d + 1\right) - B\left(\frac{a}{NE[\zeta] \mp N^c}; d - \gamma, mN - d + 1\right) \right).$$

Let us denote $N^c/NE[\zeta]$ by ε , then we get

$$S_{\mp} = \frac{\gamma a^\gamma \binom{mN}{d} (1 \mp \varepsilon)^{d-\gamma}}{(NE[\zeta])^\gamma (1 \pm \varepsilon)^d} \left(B\left(\frac{\varepsilon}{2 \mp 2\varepsilon}; d - \gamma, mN - d + 1\right) - B\left(\frac{a}{NE[\zeta](1 \mp \varepsilon)}; d - \gamma, mN - d + 1\right) \right).$$

We will use the following estimates for the incomplete beta-function:

$$B(x; a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt = O\left(\int_0^x t^{a-1} dt\right) = O\left(\frac{x^a}{a}\right),$$

$$B(x; a, b) = B(a, b) - \int_x^1 (1-t)^{b-1} dt = B(a, b) + O\left(\frac{(1-x)^b}{b}\right).$$

These estimates give us

$$S_{\mp} = \frac{\gamma a^\gamma \binom{mN}{d} (1 \mp \varepsilon)^{d-\gamma}}{(NE[\zeta])^\gamma (1 \pm \varepsilon)^d} (B(d - \gamma, mN - d + 1) + O\left(\frac{(1 - \varepsilon/(2 \mp 2\varepsilon))^{mN-d+1}}{mN - d + 1}\right) + O\left(\frac{(a/NE[\zeta](1 \mp \varepsilon))^{d-\gamma}}{d - \gamma}\right)).$$

We will use the fact that $\frac{1}{B(d-\gamma, mN-d+1)} = \frac{\Gamma(mN+1-\gamma)}{\Gamma(d-\gamma)\Gamma(mN-d+1)} = O\left(\frac{(mN)^{d-\gamma}}{\Gamma(d-\gamma)}\right)$ and factor out the beta-function:

$$S_{\mp} = \frac{\gamma a^\gamma \binom{mN}{d}}{(NE[\zeta])^\gamma} B(d - \gamma, mN - d + 1) \frac{(1 \mp \varepsilon)^{d-\gamma}}{(1 \pm \varepsilon)^d} \times \left(1 + O\left(\frac{(1 - \varepsilon/(2 \mp 2\varepsilon))^{mN-d+1} (mN)^{d-\gamma}}{\Gamma(d - \gamma)(mN - d + 1)}\right) + O\left(\frac{(am/NE[\zeta](1 \mp \varepsilon))^{d-\gamma}}{\Gamma(d - \gamma + 1)}\right) \right). \quad (15)$$

Recall that $\varepsilon = N^{c-1}/E\xi$. Let us simplify Equation (15):

1.

$$\begin{aligned} & \frac{\gamma a^\gamma \binom{mN}{d}}{(NE[\xi])^\gamma} B(d - \gamma, mN - d + 1) \\ &= \frac{\gamma a^\gamma}{(NE[\xi])^\gamma} \frac{\Gamma(mN + 1)}{\Gamma(d + 1)\Gamma(mN - d + 1)} \frac{\Gamma(d - \gamma)\Gamma(mN - d + 1)}{\Gamma(mN + 1 - \gamma)} \\ &= \frac{\gamma}{d^{\gamma+1}} \left(\frac{am}{E[\xi]}\right)^\gamma (1 + o(1)) \end{aligned}$$

if d and N grow.

2.

$$\frac{(1 \mp \varepsilon)^{d-\gamma}}{(1 \pm \varepsilon)^d} = \frac{(1 \mp N^{c-1}/E\xi)^{d-\gamma}}{(1 \pm N^{c-1}/E\xi)^d} = 1 + o(1)$$

if $d = o(N^{1-c})$.

3.

$$\begin{aligned} & O\left(\frac{(1 - \varepsilon/(2 \mp 2\varepsilon))^{mN-d+1} (mN)^{d-\gamma}}{\Gamma(d - \gamma)(mN - d + 1)}\right) \\ &= O\left(\frac{e^{(mN-d+1) \log(1 - N^{c-1}/(2E[\xi] \mp 2N^{c-1})) + (d-\gamma-1) \log(mN)}}{\Gamma(d - \gamma)(1 - (d - 1)/mN)}\right) \\ &= O\left(\frac{e^{\frac{-mN^c(1-(d-1)/mN)}{2E\xi \mp 2N^{c-1}} + (d-\gamma-1) \log(mN)}}{\Gamma(d - \gamma)(1 - (d - 1)/mN)}\right) \\ &= O\left(\frac{e^{\frac{-mN^c(1-o(1))}{2E\xi \mp o(1)} + d \log(mN)}}{\Gamma(d - \gamma)(1 - o(1))}\right) = o(1) \end{aligned}$$

if $d < mN^c/2E\xi \log(mN)$ which is true for sufficiently large N as soon as $d = o(N^{1-c})$ and $1/2 < c < 1$.

4. $O((am/E[\xi])(1 \mp \varepsilon))^{d-\gamma}/\Gamma(d - \gamma + 1) = o(1)$ if d and N grow.

Thus, we first demonstrated that the main term in (15) is equal to $(\gamma/d^{\gamma+1})(am/E[\xi])^\gamma (1 + o(1))$. Then, we showed that the error multiplier equals $1 + o(1)$. Finally, we proved that two error summands are equal to $o(1)$. To conclude the proof of the lemma it remains to note that $E[\xi] = \gamma a/(\gamma - 1)$, therefore $(\gamma/d^{\gamma+1})(am/E[\xi])^\gamma = (\gamma/d^{\gamma+1})(\gamma - 1)m/\gamma^\gamma$. \square

Proof of Lemma 3

First, recall the statement of Lemma 3 from Section 5.1.3.

LEMMA 3 Assume that both d and N grow, $d = o(N^{1-c})$, $d = o(e^{\varphi(N)})$ and $q \leq N^r$, then

$$S_{\mp}(d, q) = \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma - 1)m}{\gamma} \right)^{\gamma} (1 + o(1)).$$

Proof. Recall that

$$S_{\mp}(d, q) := \int_a^{N^r} f(q) \rho_{\mp}(d, q) dq,$$

where

$$\rho_{\mp}(d, q) = \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{q e^{-[i/m]/N}}{NE[\zeta] \pm N^c} \right) \sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} \prod_{j=1}^d \frac{q e^{-[i_j/m]/N} / (NE[\zeta] \mp N^c)}{1 - q e^{-[i_j/m]/N} / (NE[\zeta] \pm N^c)}. \quad (16)$$

Therefore, we first prove the following lemma on the behaviour of $\rho_{\mp}(d, q)$. □

LEMMA 4 Under the condition of Lemma 3 we have

$$\rho_{\mp}(d, q) = \left(1 + O\left(\frac{q^2}{N}\right) + o(1) \right) \left(\frac{qm}{E[\zeta]} \right)^d \frac{e^{-qm/E[\zeta]}}{d!}.$$

Proof. Note that

$$\begin{aligned} \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{q e^{-[i/m]/N}}{NE[\zeta] \pm N^c} \right) &= \prod_{i=0}^{m\varphi(N)N} \left(1 - \frac{q e^{-i/mN} e^{O(1/N)}}{NE[\zeta] \pm N^c} \right) \\ &= \exp \left(\sum_{i=0}^{m\varphi(N)N} \log \left(1 - \frac{q e^{(-i/m)/N} (1 + O(1/N))}{NE[\zeta] \pm N^c} \right) \right) \\ &= \exp \left(- \sum_{i=0}^{m\varphi(N)N} \left(\frac{q e^{-i/mN} (1 + O(1/N))}{NE[\zeta] \pm N^c} + O\left(\frac{q^2 e^{-2i/mN}}{N^2}\right) \right) \right) \\ &= \exp \left(- \frac{q (1 - e^{-\varphi(N)})}{(1 - e^{-1/mN})(NE[\zeta] \pm N^c)} \right. \\ &\quad \left. + O\left(\frac{q (1 - e^{-\varphi(N)})}{N^2(1 - e^{-1/mN})}\right) + O\left(\frac{q^2 (1 - e^{-2\varphi(N)})}{N^2(1 - e^{-2/mN})}\right) \right) \\ &= \left(1 + O\left(\frac{q^2}{N}\right) + O(N^{c-1}) + O(e^{-\varphi(N)}) \right) \exp\left(\frac{-qm}{E[\zeta]}\right) \\ &= (1 + o(1)) \exp\left(\frac{-qm}{E[\zeta]}\right). \end{aligned} \quad (17)$$

Here we used the fact that $q \leq N^r < N^{1/2}$ since it allows us to estimate $\exp(O(q^2/N))$ as $1 + O(q^2/N)$.

Let us continue

$$\prod_{j=1}^d \frac{q}{(1 - q e^{-[i_j/m]/N} / (NE[\xi] \mp N^c)) (NE[\xi] \mp N^c)}$$

$$= \left(\frac{q}{NE[\xi]} \right)^d \left(1 + O\left(\frac{dq}{N}\right) + O(dN^{c-1}) \right) = \left(\frac{q}{NE[\xi]} \right)^d (1 + o(1)), \tag{18}$$

$$\sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} \prod_{j=1}^d e^{-[i_j/m]/N} = \sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} e^{(-i_1 - \dots - i_d)/mN} \left(1 + O\left(\frac{d}{N}\right) \right)$$

$$= \sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} e^{(-i_1 - \dots - i_d)/mN} (1 + o(1)). \tag{19}$$

It remains to estimate $\sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} e^{(-i_1 - \dots - i_d)/mN}$. We use the following notation:

$$F(k, d) = \sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} e^{(-i_1 - \dots - i_{d-1} - ki_d)/mN}. \quad \square$$

LEMMA 5 If $d(k + d) = o(N)$ and $k + d = o(e^{\varphi(N)})$, then

$$F(k, d) = \frac{(mN)^d (k - 1)!}{(k + d - 1)!} (1 + o(1)).$$

Proof. Note that

$$F(k, 1) = \sum_{0 \leq i_1 \leq m\varphi(N)N} e^{-ki_1/mN} = \frac{1 - e^{-k\varphi(N) - k/mN}}{1 - e^{-k/mN}}.$$

Let us get a recurrent formula for $F(k, d)$:

$$F(k, d) = \sum_{0 \leq i_1 < \dots < i_d \leq m\varphi(N)N} e^{(-i_1 - \dots - i_{d-1} - ki_d)/mN}$$

$$= \sum_{0 \leq i_1 < \dots < i_{d-1} \leq m\varphi(N)N} e^{(-i_1 - \dots - i_{d-1})/mN} \sum_{i_d=i_{d-1}+1}^{m\varphi(N)N} e^{-ki_d/mN}$$

$$= \sum_{0 \leq i_1 < \dots < i_{d-1} \leq m\varphi(N)N} e^{(-i_1 - \dots - i_{d-1})/mN} \frac{e^{-k(i_{d-1}+1)/mN} - e^{-k\varphi(N) - k/mN}}{1 - e^{-k/mN}}$$

$$= \frac{e^{-k/mN}}{1 - e^{-k/mN}} \sum_{0 \leq i_1 < \dots < i_{d-1} \leq N\varphi(N)} (e^{(-i_1 - \dots - (k+1)i_{d-1})/mN} - e^{-k\varphi(N)} e^{(-i_1 - \dots - i_{d-1})/mN})$$

$$= \frac{e^{-k/mN}}{1 - e^{-k/mN}} (F(k + 1, d - 1) - e^{-k\varphi(N)} F(1, d - 1)).$$

It is easy to get an upper bound for $F(k, d)$

$$\begin{aligned} F(k, d) &\leq \frac{e^{-k/mN}}{1 - e^{-k/mN}} F(k + 1, d - 1) \leq \dots \\ &\leq \frac{e^{-(2k+d-2)(d-1)/2mN} F(k + d - 1, 1)}{(1 - e^{-k/mN}) \dots (1 - e^{-(k+d-2)/mN})} \leq \frac{e^{-(2k+d-2)(d-1)/2mN}}{(1 - e^{-k/mN}) \dots (1 - e^{-(k+d-1)/mN})} \\ &= \frac{e^{-(2k+d-2)(d-1)/2mN}}{(1 - e^{-k/mN}) \dots (1 - e^{-(k+d-1)/mN})} = \frac{(mN)^d (k - 1)!}{(k + d - 1)!} \left(1 + O\left(\frac{(k + d)d}{N}\right) \right). \end{aligned}$$

Using this upper bound and the recurrent formula above we can find a lower bound. Assume that

$$\begin{aligned} F(k, d) &= \frac{e^{-k/mN}}{1 - e^{-k/mN}} (F(k + 1, d - 1) - e^{-k\varphi(N)} F(1, d - 1)) = \dots \\ &= \frac{e^{-(2k+d-2)(d-1)/2mN} F(k + d - 1, 1)}{(1 - e^{-k/mN}) \dots (1 - e^{-(k+d-2)/mN})} - \sum_{i=1}^{d-1} \frac{e^{-(2k+i-1)i/2mN} e^{-(k+i-1)\varphi(N)} F(1, d - i)}{(1 - e^{-k/mN}) \dots (1 - e^{-(k+i-1)/mN})} \\ &= \frac{(mN)^d (k - 1)!}{(k + d - 1)!} \left(1 + O\left(\frac{(k + d)d}{N}\right) \right) \\ &\quad - \sum_{i=1}^{d-1} e^{-(k+i-1)\varphi(N)} \frac{(mN)^{d-i} (mN)^i (k - 1)!}{(d - i)! (k + i - 1)!} \\ &\quad \times \left(1 + O\left(\frac{(d - i)^2}{N}\right) + O\left(\frac{(k + i)i}{N}\right) \right) \\ &= \frac{(mN)^d (k - 1)!}{(k + d - 1)!} \left(1 + o(1) - \sum_{i=1}^{d-1} \frac{e^{-(k+i-1)\varphi(N)} (k + d - 1)!}{(d - i)! (k + i - 1)!} (1 + o(1)) \right) \\ &\geq \frac{(mN)^d (k - 1)!}{(k + d - 1)!} \left(1 + o(1) - (1 + o(1)) \sum_{i=1}^{d-1} \frac{((k + d - 1)e^{-\varphi(N)})^{k+i-1}}{(k + i - 1)!} \right) \\ &= \frac{(mN)^d (k - 1)!}{(k + d - 1)!} \left(1 + o(1) + O\left(\frac{((k + d - 1)e^{-\varphi(N)})^k}{k!}\right) \right) \\ &= \frac{(mN)^d (k - 1)!}{(k + d - 1)!} (1 + o(1)). \end{aligned}$$

Finally, taking into account Equations (16–19) and Lemma 5, we get

$$\begin{aligned} \rho_{\mp}(d, q) &= e^{-qm/E[\xi]} \left(\frac{q}{NE[\xi]} \right)^d F(1, d) (1 + o(1)) \\ &= e^{-qm/E[\xi]} \frac{(mN)^d}{d!} \left(\frac{q}{NE[\xi]} \right)^d (1 + o(1)) = (1 + o(1)) \left(\frac{qm}{E[\xi]} \right)^d \frac{e^{-qm/E[\xi]}}{d!}. \end{aligned}$$

Now we can estimate

$$S_{\mp}(d, q) = \int_a^{N^r} f(q) \rho_{\mp}(d, q) \, dq = \int_a^{N^r} (1 + o(1)) \frac{\gamma a^{\gamma}}{q^{\gamma+1}} \left(\frac{qm}{\mathbb{E}[\zeta]} \right)^d \frac{e^{-qm/\mathbb{E}[\zeta]}}{d!} \, dq.$$

We get an incomplete gamma function:

$$\begin{aligned} S_{\mp}(d, q) &= (1 + o(1)) \frac{\gamma a^{\gamma} m^{\gamma}}{d! (\mathbb{E}[\zeta])^{\gamma}} \int_{am/\mathbb{E}[\zeta]}^{N^r m/\mathbb{E}[\zeta]} x^{d-\gamma-1} e^{-x} \, dx \\ &= \frac{\gamma(1 + o(1))}{\Gamma(d+1)} \left(\frac{am}{\mathbb{E}[\zeta]} \right)^{\gamma} \left(\Gamma\left(d - \gamma, \frac{am}{\mathbb{E}[\zeta]}\right) - \Gamma\left(d - \gamma, \frac{N^r m}{\mathbb{E}[\zeta]}\right) \right) \\ &= (1 + o(1)) \frac{\gamma \Gamma(d - \gamma)}{\Gamma(d + 1)} \left(\frac{am}{\mathbb{E}[\zeta]} \right)^{\gamma} = \frac{\gamma}{d^{\gamma+1}} \left(\frac{am}{\mathbb{E}[\zeta]} \right)^{\gamma} (1 + o(1)) \\ &= \frac{\gamma}{d^{\gamma+1}} \left(\frac{(\gamma - 1)m}{\gamma} \right)^{\gamma} (1 + o(1)). \end{aligned}$$

This concludes the proof. □